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PROBABILITY (OF DEFAULT) DOES NOT EXIST

Speech on Simplicity and Comparability of the Risk Measures

**Abstract**

“The probability: who is it? Before answering [...] it is certainly worthwhile to ask yourself: does probability really exist? And what would it be?” wrote Bruno de Finetti in the item *Probability* of the “Encyclopaedia Einaudi”. The probability is our guide in thinking and acting under uncertainty. In banking environments – with Basel 2 – it has acquired an institutional role. The probability (of default) influences the bank’s capital, the profitability and the convenience of the business and, at the end, the amount of the credit towards the real economy. If on the one hand much time and care were spent in a progressive refinement of the analytical framework for its numerical evaluation, on the other hand, minor attention was pay to the basic concepts related to the “foundation” of the probability. Many disputes in the international working groups are still spoiled by serious misunderstandings which makes difficult the solution of the technical problems. Still today we see proposals and suggestions – de Finetti would have said *superstitions* – which a rigorous debate has definitively filed. This work is an attempt to repossess – in the simpler style, already from its design, in the form of *Dialogue* – the cornerstones of reasoning (and action)

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*Probability (of Default) Does Not Exist...*

under uncertainty, in the assumption of the indissolubility between theory and practice, in the strong belief that many practical problems come from lacking knowledge, or misunderstanding, of deep theoretical concepts.

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## 1. Introduction

This work was published – in Italian language – as “Quaderno Minerva Bancaria” no. 6, with the Italian title *La probabilità (di default) non esiste. Discorso sopra la comparabilità delle misure di rischio*.

The style of the *Dialogue* was inspired – with temerity, we admit it – to the *Dialogo sopra i due massimi sistemi del mondo*, of Galileo Galilei.

The name “Nottola” wants to remind Minerva’s owl.

The name “Chiara” evokes the desire to make simple the risk measurement processes, a feature often requested in the banking environment, although it is tinged with of ingenuity when the discussion moves from the theoretical principles to the methods of implementation.

The phrases of “Bruno” came from to the writings of Bruno de Finetti, with a stylistic freedom justified only by the desire to maximize the effectiveness of the arguments. We have assigned to “Bruno” also some phrases of other authors. However, they are always quoted by de Finetti in his writing as well as they are aligned (in whole or in part) with de Finetti’s point of view. In other cases, we have assigned to “Bruno” some our phrases, which are anyway heavily influenced by reading de Finetti’s books.

We have decided to loot the writings of de Finetti because we believe that his view on the uncertainty, risks, opinions and accountability should become a mainstream, when people discuss about practical problems of risk management, in bank or in insurance company, as well as from a supervisor’s perspective. Furthermore, we have to consider the fundamental behavioural and cognitive aspects that recent experiments have revealed constructively in the past three decades.

We could not say better what had already been perfectly said by an intellect not able to be compared to ours, although de Finetti talk about probability in a wide sense, whereas we use it – every day, with huge difficulties of communication – in a technical field, the measurement of banking risks, on which we have reformulated some of his thoughts. We have the strong belief that the pragmatic writings of de Finetti – for the relevance of the technical contents, coloured with a brilliant phrases, filled with evocative images, if necessary enlivened by a sophisticated irony – are the best antidote against metaphysical tendencies in the discussions about the risk measurement in bank or insurance companies.

It is a standard – in the official publications – signpost in the text the references from which the quotations are resumed. In our case, however, this standard would unduly burden the *Dialogue* and the reader would be overwhelmed by referrals to the bibliography. We therefore prefer simply to list the texts used, leaving the reader the pleasure of discovering the contents and the quotes.

## **2. A dialogue on risk measurement**

*There were ladies and gentlemen, who wanted to satisfy their thirst for learning. So they decided to meet, to talk about uncertainty and risks as well as their effects and ways to measure them. After the debts and short compliments, Nottola so began ...*

**Nottola:** *Basel 2* has introduced into the practice of the banking business a powerful math, to make the pricing of the contracts, to evaluate the balance sheet, to fix the capital requirement; it has given an institutional role to probability distributions (on the future); it has reformed the organizational profile of firms and authorities, changing the responsibilities and the day-by-day activities. Along with math and statistics techniques, were required an appropriate IT infrastructure, the legal culture and even some tax considerations. Risk management – with *Basel 2* – has become a general problem of “corporate governance”, which imposed to define, or redefine and rebuild, a non-trivial relationship between interdisciplinary and specialization. The enthusiasm of the long design phase, however, was paradoxically replaced by a progressive distrust precisely when the new regulatory framework should have been applied, for the unexpected changes in market scenarios due to the evolution of the financial crisis. What do you think about *Basel 2*, whereas now many people pay attention to the *Basel 3* and we begin to glimpse a version number four?

**Chiara:** I want to be fair. The progress is marked by human errors, which have become as many sources of learning. Bankruptcies and the ability to absorb shocks, to use them as a staircase, qualify the rational action of each of us.

**Nottola:** What are your doubts?

**Chiara:** With *Basel 2*, through internal models, we have only the opinions of the risk managers, rather than actual information, reliable, verifiable, and objective, on the soundness of banks. The accuracy of the math in *Basel 2* is only an illusion. The underlying math in *Basel 2* is in truth mechanistic and

reductionist facts. The supervisors have become captives of their own instruments.

**Bruno:** It is ironic! You are in the prison of the prejudice, but you think that the captives are who look inside through the bars. You mocks Basel 2, for his honesty to analyse the banking risks, and you regret to have no more available a standardized methods, which are equivalent to the adoption of an arbitrary opinion, with the dubious merit to be the opinion of an anonymous Mr. Somebody, that you prefer just to avoid responsibility of a choice. Those empirical *escamotage* for the calculation of capital requirements – those *adhockeries*, *ad hoc* solutions devoid of a single guiding principle – seem to be free to subjective judgments, not because they really avoided them, but simply because they do not nominate them. In this way, you do not restrict them and you leave them the whole field!

**Chiara:** Basel 2 was an unforgivable optimism momentum for those who, by institutional mission, should always make careful and act prudently: the capital of the bank - with Basel 2 - depends on the probability of default of the counterparties ...

**Bruno:** ... and what else you should expect, if not a linkage between the capital level and the risk of the counterparties, as measured by the probability of default?

**Chiara:** ... with Basel 2, if you let me finish, each bank is free to evaluate the odds as it wishes ...

**Bruno:** ... not as the bank wants, but through a model, in order to have the consistency in the evaluation, under to the judgment of the Supervision Authority ...

**Chiara:** ... which still does not ensure the same probability of default for the same counterparty, if the evaluation is performed by different banks!

**Nottola:** Why different banks should assign the same probability to the same counterparty? What is relevant to the probability assessment? The *uniqueness* of the counterparty or the *diversity* of the banks?

**Chiara:** I do not want to deal pedantic and obscure discussions.

**Bruno:** Not even me. The verbal acrobatics do not enjoy me and I am bored to besiege castles of words built on the clouds.

**Nottola:** If we want to develop a reliable reasoning, then we must know exactly what we are discussing. Everyone talk about probability, but nobody is able explain the meaning of the probability in a satisfactory way. I would like to understand the concepts underlying to the probability of default and I

would know whether it is reasonable to expect assessments aligned by several banks.

**Bruno:** Giovanni Vailati – an Italian philosopher and mathematician – argued that the worst way to make sure of the degree of knowledge that an individual has of a concept, is to ask him what the concept is *per se*. I am too pragmatic to worry about the *noumena*. Faced with a concept, whatever it is, I analyse the deep and essential motives that formed the purpose for which that concept was introduced, the reason that explain the intimate reason for its usefulness.

**Nottola:** I revising my question: which is *the scope* of the probability?

**Bruno:** The scope of the probability is clarify, to each of us, the own degree of doubt about an uncertain event, depending on the partial information which we have and depending on the algorithm used to process them.

**Chiara:** So?

**Bruno:** As consequence, the probability is not *a property of the event* but it is *a relationship between the event and a person* who assesses its likelihood. It does not therefore have an autonomous existence, separable from the person that judges it. The probability does not exist, if not for me, depending on the degree of ignorance in which I am. There is no chance, outside the evaluations that can be done with reflection or thought the instinct. And if you assessed the probabilities with the utmost care, as if they were objective, but at the same time with the greatest sense of responsibility to avoid self-deception to believe that probabilities are really objective, then there is no reason to doubt that my evaluation is wrong, because it does not make any sense outside to me, it has no other purpose than to express what my state of mind is.

**Chiara:** *A state of mind?* If you want to give to the probability an institutional role, if you want to consign to the probability a crucial task to bank regulation purposes, then it should be an objective probability, otherwise your target is not feasible.

**Bruno:** You talk about of the probability of default as if it was a physical attribute of the debtor, tangible but not observable. A borrower will have a height, a weight, an income, a civil state, and you believe that he can have in the same sense a probability of default, objective as are height, weight, marital status and income, even though not immediately visible. You see in the probability a sort of dark number, with a physical meaning, or it would be better to say metaphysical, to be placed between the *Planck mass* and

*Bohr magneton*, a mysterious mechanism which rule the happen or otherwise of an event.

**Chiara:** The banks hallmark their counterpart's whit the probability of default. Depending on this number, we have different capital requirement, different prices for the borrowers, different accounting results, and as consequence a difference in the balance sheet and in the income statement, which are real and objective.

**Bruno:** That the accounting figures are true, real and objective, is a exaggerate point of view. There is no truth, unique and unambiguous, about the balance sheet. Given the same material things in the assets and liabilities sides, the truth of a balance sheet constructed for the purpose of liquidation is a different truth from that of the balance sheet built to meet the dividend to be distributed to shareholders; and the truths of the budget of a firm where the shareholders are one or are the members of a family or a stable group of people, or still many changeable and unstable people, are many different truths, but all are fair. If you change the aim of the evaluation, the criteria to assess the firm will change accordingly.

**Chiara:** This his opinion only creates confusion with respect to our discussion.

**Bruno:** To be honest, it is an opinion of Luigi Einaudi and I agree with him, because this view is useful also to understand what precautions and what precautions will be able to make propositions with sense, in terms of probability. Affinity is perfect and even goes beyond my intentions. You desire on the comparability of risk-weighted assets, as if the values of the assets, pure and simple, were objective.

**Chiara:** Dividends paid to shareholders and bonuses received by managers are true and real; they depend on the balance sheet, which we should presume reliable, in turn influenced by the probability of default, ... *for which you want to deny all objectivity!* If the probabilities are not objective then what are they?

**Bruno:** I repeat myself. The probability is a mood and you should include it among the psychological feelings: the probability is an *instinct obscur*, *dont nous ne pouvons nous passer*, in the evocative words of Poincare; it is *the expression of our spirit in a state of imperfect knowledge*, the clear image of De Morgan; it is a *reality of the mind*, in the concise statement of Havvelmo; it is – I say – the opinion of a person about of the plausibility of an event (for him) still uncertain, translated in number and subject to consistency constraints.

**Chiara:** We are discussing about capital of the banks! No philosophy, please.

**Nottola:** What are the conclusions of this argument? If Bank X lend 10,000 euros to Miss Truth and evaluates her probability of default to 3%, then it set aside approximately 300 EUR, to hedge against the risk of default. And even if the Bank Y lend 10,000 euros to Miss Truth, but it assesses the probability of default to 1%, then it should set aside only 100 euros?

**Chiara:** Rather embarrassing, and impossible to justify, if not venturing into a labyrinth of sophistries. Miss Truth is always Miss Truth, one and only, with its probability of default. We can't accept a regulatory scheme incapable of seeing the non-sense of two banks that assign different default probabilities to the same counterpart, with all the dramatic consequences for the supervision activities. It is unexplainable as the Supervision Authority, always oriented to urge the uniformity of judgments in front the same facts – to name one, the concept of *sofferenza* – can live together with a *pot-pourri* of assessments.

**Bruno:** I read in the official documents that *sofferenza* refers to “counterparties in a state of insolvency, even if not legally ascertained, or in similar situations, regardless of any loss forecasts formulated by the bank”. For the *sofferenza* the dreaded event has occurred, the risk of default no longer exists, so the probability of default no longer makes sense, and the purpose of the *sofferenza rettificata* is to make the information available to those who still are not aware about them. The situation is quite different, when we talk of uncertainty and probability.

**Nottola:** Even the probability depends on the information...

**Bruno:** ... with a clarification, though. To speak properly about probability you have to know something and ignore something else, because if you know all, then the uncertainty vanish and the probability collapse to 0 or 1, the only true values in a strict sense, referring to two extreme and pleonastic cases, in which the idea of probability is no longer useful. If we are not ignorant, then there no probability! The concept was already clarified by Jacob Bernoulli: *probabilitas est gradus certitudinis et ab hac differet ut pars in toto*, to suggest a quantitative assessment of knowledge in the psychological premise of their limits. Then again Keynes: every proposition is true or false, but our knowledge in this regard depends on the circumstances.

**Nottola:** What do you want to proof?



**Bruno:** The judgment of the probability of the same event can change from one person to another – from one bank to another, if we are interested to credit risk – because are different the respective partial information.

**Chiara:** We could imagine a set of information, which although incomplete, is favourite with respect to the assessment of the probability.

**Bruno:** The only privilege would be known all, do not have more uncertainties, and therefore make less of the probability. But in every concrete situation of uncertainty, the degree of informational incompleteness is a matter of the problem and we must learn to live with it. You may decide to acquire new information and then you will have a new probability, that does not contradict the previous probability, but it simply passes from one thing (the probability in a certain state of information) to another (the probability in another state of information) which is different from the previous one to be in agreement with it. If I say “my watch marks 11.25”, I do not correct the assertion made an hour ago, “my watch marks 11.25”, but I modify it to preserve the coherence.

**Chiara:** We could at least ensure that all banks have the same set of information, in assessing the probability of default.

**Bruno:** The banking business is based on private information, the imbalances of knowledge are unavoidable and they qualify advantages and disadvantages in the market. No bank would be willing to share its information set. On the other hand, if you want to share common information and then oblige banks to make a probability assessment subordinated just common knowledge – a sort of intersection of information available in the banking system – you would get to the undesirable situation of a measure calibrated on the smallest set of information. However, even by omitting business considerations – at the end I am neither an economist nor a banker – there is a discourse that I care most.

**Nottola:** What?

**Bruno:** A hypothetical coincidence of set of information, if can facilitate the uniform judgments, does not guarantee it. The information sharing does not bind to a convergence of *views*, because everyone is still free to be understood in its own way the different parts of the same body of knowledge. It would be a blessing, if when we talk about the probability of an event  $E$ , we keep in mind the writing  $\mathbf{P}_I(E|H)$ , where is clear the dependence of the probability  $\mathbf{P}$  from the set of information  $H$  as well as from the individual  $I$  that judges, instead the simplistic notation  $\mathbf{P}(E)$ , which brings up the

probabilities as something eternal and immutable, like the smile of the Cheshire Cat, which remains even after the cat went away.

**Nottola:** We could establish at least fix a boundary, an upper and a lower limit, a range within which a probability assessment must fall, to be reasonable.

**Bruno:** I do not agree, because, instead of deleting the embarrassment of choosing a precise value for the probability, doubles it, forcing you to choose two precise boundaries.

**Chiara:** If everything can be true, when we talk about risk-measurement, then this confirms the need to rethink the regulation. A regulation framework based on opinions does not work. It serves objectivity, at times even conventional parameters, but based set on objective facts.

**Bruno:** The importance of knowing the facts, to know them in the form of numerical statistical data, to have a mentality in which they occupy a crucial position, is out of the question. *But a fact is like a sack: blank does not hold.* probability with the facts as you do a house with stones, but a bunch of facts is not a probability like a bunch of stones is not a home. You can gather factual data as judgments, but the answer is not in fact. It remains in subjective *opinion*, that no fact can constrain, but that certain facts may possibly spontaneously and feel constrained.

**Chiara:** I cannot accept this point of view ...

**Bruno:** Why? Every day we can see paradoxical conflicts, in which, in support of opposing arguments, the parties invoke the same facts – on which they are agree, sometimes even in the smallest details – but from which they draw different conclusions.

**Chiara:** Please, we are not discussing about ordinary everyday problems, where everyone is free to decide. Here we talk of the capital of banks, although the frequent digressions seem to aim at confusing.

**Bruno:** No digression from my side. I just do not believe in a way of thinking for the day-to-day decisions and in a different way for the technical, scientific, or institutional issues. I think, by contrast, that there is simply a unified concept for the things of the world, very good for all applications that can be the bets on sport events or the decisions about capital levels of the banking systems.

**Nottola:** But *in practice* what should we do when faced with a discrepancy of probabilistic assessments of the same event?

**Bruno:** Knowing that different banks set a different value for the probability of one event is a fact that should be used, but not to mediate

between evaluations, hoping in the miraculous *in medium stat virtus*, or to force the lowest value to the highest, in an illusory exercise of conservativeness. We must use this fact to better understand the problem, in terms of information and models used.

**Nottola:** We see an immense task for supervisors. Are we sure that they are capable?

**Bruno:** I do not know, but *people had no choice*, as Robert Merton commented on the inevitability of the use of Black-Scholes formula, although it was made of esoteric concepts, outside to the curricula of market participants, even if then they began to speak of hedge ratio and delta, to appoint the lemma Itô and differential equations.

**Chiara:** Philosophy, philosophy! We will see if it is not possible to give the probability an objective meaning. When you bet on the launch of a coin or a dice, when you play roulette or betting, you have an objective probability of win. In switching from the case of the games, *per se* trivial, to more relevant uncertainties, there is no reason to think that there are not the same objective probabilities.

**Bruno:** You must keep me apologized, but *I do not understand*: what does it mean that the probability of “head and cross” is objectively the 50%?

**Chiara:** I do not care about coin. I quoted the paradigm “head and cross” to describe a situation where you have a true probability, 50 and 50, and I do not see why it should not be so elsewhere.

**Bruno:** The idea of a true probability is an aberrant superstition and it is most disappointing thing to hear again - presented as critical - so superficial clichés. I do not understand if who believes in certain misconceptions has read the arguments to refute them or if they have read them without understanding anything or instead if they have understood them in reverse.

**Chiara:** If you want to make it clear also for us ...

**Bruno:** You talk about of probability of the coin as if it were the probability inside the coin, melted in the coin in a physical sense, in one with metal alloys from which it is composed, as well as you speak of the probability of default as if it were a genetic item of the counterparty. It seems to me to do the utmost to understand the discussions of others, to respond carefully and patiently, even when it comes to repeating things that have been said and repeated, in order to argue trivial misunderstandings. Rarely I have the pleasure of having the impression that others make a similar effort.

**Nottola:** I invite you to make an effort again ...

**Bruno:** The coin has a shape, has a diameter, has a thickness, has a weight, but it cannot be said, in the same way, which has a probability. The Polygraph, among the technical specifications of the mintage, certainly does not take into account the probability of the two faces. The probability is not – it cannot be, even to want – a physical characteristic of the coin. Many people love to say “perfect coin”, in their discussion on the probability, but nobody can formalize this concept.

**Chiara:** But in the case of the coin, all agree in assessing the probability 50 and 50.

**Bruno:** Because everyone is in the same state of mind. I estimate the probability to 50% because I feel a symmetrical uncertainty respect to the two faces, because I cannot find reasons to give priority to the output of one on the other face, because to be perfect or symmetrical is my ignorance in front of the outcome of the launch, not the coin itself. Then, the fact of experiencing a universally shared mood, the fact that everyone evaluates in 50 and 50 the probability of "head and cross" does not transform it into something different from my mood, does not make it objective simply because unanimous. If you think to reach objective conclusions, by layering the subjective opinions concordant, is like believing that a heap of stones, increasing more and more, at the end will eventually become an animal.

**Chiara:** When you extract the balls by urn - and forgive me if I insist on school cases, but it is only to show clearly the general idea - when you extract the balls, I said, the chance of having a red or a yellow ball is objectively determined by their proportion within the urn.

**Nottola:** So, if the urn contains 50 red and 50 yellow balls, the probability is objectively 50% for each colour and if you believe in a different probability, then you are wrong.

**Chiara:** Yes. Of course, the urn is merely a trick, but it broadly defines a general schema, useful to understand the problems, in the areas of actual interest.

**Bruno:** Imagine then having to assign a judgment to the probabilities evaluated by three individuals about the extractions from an urn, in the same way that a supervisor is called upon to express himself or herself on the quality of the probability of default assessed by different banks. The first individual declares a probability of 50%. Is an acceptable judgment?

**Chiara:** It is the only acceptable one. An alternative assessment would be extravagant, incomprehensible, without justification.

**Bruno:** The second individual, however, declares a 40% probability for the exit of a red ball.

**Chiara:** Unacceptable!

**Bruno:** Why?

**Chiara:** The urn contains red and yellow balls in equal proportion, 50 and 50, randomly shuffled, so ...

**Bruno:** I know, I know, .... Now we are addressing the third person, who says with confidence 99%.

**Chiara:** A crazy!

**Bruno:** However, we are curious to know his justification. Therefore, we approach him to ask explanations and we realize that from its position, from his viewing angle, from his perspective, thanks to a game of mirrors, he able to discern clearly the urn content. Now we also see what he sees. The random shuffling of the balls has given rise to a rather curious arrangement: all fifty reds up, the fifty yellow down. Is his 99% probability still a madness?

**Chiara:** Not anymore.

**Bruno:** Now is it the first individual who stated 50% to make a mistake?

**Chiara:** Yes!

**Bruno:** Please, let us emend him!

**Nottola:** ... but we should give to him the private information of third party; otherwise, he will not agree to deviate from the most natural evaluation.

**Bruno:** Please, no scruples of confidentiality, faced with the possibility to know and spread the true probability! We surely go to the first person, whispering in his ear the particular arrangement of balls known to the third.

**Nottola:** Now the assessments will be the same ...

**Bruno:** ... no, unfortunately, because once extended its set of knowledge, the first individual with equal confidence answers 1%!

**Nottola:** I do not understand ...

**Bruno:** Simply, he knows the typical behaviour of who extract the balls, to have repeatedly observed him in the past. Almost never he extract in surface. He loves instead sink the urn arm, in depth. As consequence, there are low chance to have a red ball. What do we think of now of this probability?

**Chiara:** This is the right evaluation ...

**Bruno:** Well, depending on the available of new information, it was baptized "true" a probability of 50%, of 99%, and of 1% and we don't care to

hear the reasons of the second individual, which had evaluated the probability at 40%, which may have similar arguments as well, if not more convincing than the other two, in support of its assessment. Each of us evaluate the probability with the information that he has, for each of us there is a personal probability, as Savage calls it, each in his own way, as Pirandello said, each with its own probability, no more true or false than the other, which only require consistency with other probabilities evaluated by the same individual.

**Nottola:** I am more confused than convinced ...

**Bruno:** The confusion arises because people confuse the composition of the urn, which is a physical fact, with the probability that is a psychological state dependent on information. The examples, if desired, may multiply. Facing physical states in appearance symmetric - the two sides of a coin, the six faces of the dice, the thirty-seven chance of roulette or the composition of an urn - there will always be additional information that, if known, will influence our opinion and all the physical symmetries will be destroyed with their associated probabilities.

**Nottola:** So?

**Bruno:** So, the objective probability never exist , even in the very special field of play - where we have only a convergence of opinions, due by particular conditions - and a fortiori it cannot exist in the field of banking risks, dominated by information disparities and divergence of beliefs. Each bank will have its model, built on judgments, information and motivation you will be able to know or not to know, to share or not share; you can judging it more or less reasonable - with double subjective judgment: our in comparison to the other -, and it can be more or less close to those of a few or many or all other banks. But in an evaluation of probability we can judge only if it is or is not consistent , and consistency is ensured by its use of a mathematical model, whereas if you would like evaluate the probability only with the intuition or instinct, then it's high the danger to fall in irreconcilable contradictions.

**Chiara:** I will not leave you a tooth that I consider healthy. If you launch a coin a high number of times, you will have roughly the same relative frequency of heads and crosses, and the goodness of the approximation increase with the number of the launches. The probability is objectively 50% and by analogy, out of the case of games, you can still think to determine it experimentally, through repeated observations of similar cases.

**Bruno** : *Probability and Frequency* , the female version of the Gemini of Plauto ...

**Nottola**: The Gemini of Plauto?

**Bruno**: Yes, *The Two Menecmi*, a comedy of Plauto, where two Gemini are continuously exchanged between them, arousing continuous, funny and absurd misunderstandings.

**Chiara**: What is the connection with our discussion?

**Bruno**: There are so many and complex connections between probability and frequency, two notions each other extraneous, that if you confused the one for the other, then any attempt at clarification will make the situation worse.

**Chiara**: It is all clear, from my point of view. The probability is an idealized frequency, and I am not afraid to say mythologized: it is the frequency that you should have in an infinite number of experiments, assuming they can be run in the same conditions.

**Bruno**: You present yourself as a pragmatic decision maker, but in your discussions appears the infinity, I mean situations for which no one know what will happen. In the long run ... we're all dead, even the survivors of the famous long period of Keynes. And then what does it mean repetitions performed in the *same conditions*? If the repeats were seriously carried out under the same conditions, then they would give invariably the same result, right? This confusion between probability and frequency – between probability of default and default rate, we can say – is a bad mistake in the teaching of probability theory and its persistence is paradoxically due to the fact that it is the worst interpretation of probability.

**Chiara**: The insurance companies thrive on the *frequency-probability*, their entire business is built on the idea of a frequency that approximates at the best to the probability. Insurers behave as if the frequencies were probabilities, and with these *frequencies-probability* they calculate the premiums, set the technical provisions, and fix the own funds as well as pay the dividends. If things work out, thanks to the *probability-frequencies*, cannot be just their feeling, we must have under an objective reality.

**Bruno**: Well said! It is *as if*, it is the *als ob* of Vaihinger. Why if something works in practice, do you want to see the philosophical substratum of an absolute truth?

**Chiara**: The frequencies are objective ...

**Bruno**: I do not agree. To perform the frequency calculation you have to group the single observations into the clusters, because only inside a class

you can count the “successes”, to relate to the total, to have frequency; and grouping criteria are subjective, they only have practical value and no logical necessity; any attempt to use them to have an objective definition of probability is illusory. Two objects can always bring together in the same class or separate into distinct classes, and among the many classes there will be some subjectively interesting, for which it will judge useful to introduce a special name. But it is only a question of utility and any philosophical discussion would be trivial. There is no need to seek the truth, we ask only to become aware of our own opinion.

**Chiara:** The probability of death depends objectively on the age of the insured, or do you want to deny it?

**Bruno:** I do not deny it, although I would prefer to speak of broad inter-subjective agreement. However, I admit of course a strong dependence – and I can say objective, if it can give pleasure – between age and the risk of death. So, given a population of  $N$  individuals of age  $x$ , of which  $n$  are dead, can we say that  $n/N$  is the objective probability?

**Chiara:** If the number  $N$  of the initial population is high enough ...

**Bruno:** Let us imagine that it is. Then  $n/N$  is the best estimate for the unknown probability?

**Nottola:** Not really. Age is a critical risk-driver, but the statistics says also that the life expectation of women is higher than men's, so the  $N$  units should be classified by age and sex.

**Bruno:** And when we have grouped by age and sex, at least we got *the best estimate of the true unknown probability*, as someone likes to say, with an expression of which you should suspect only for its length.

**Nottola:** Not really. There are so many other factors to consider, if we want to be precise.

**Bruno:** We want it ...

**Nottola:** The nationality, if necessary qualified for the region where you live; the marital status and eating habits; it's relevant to know if you practice or not some sports, unless it is extreme sports, which would be an aggravating factor; indeed, now that I reflect on this point, even ordinary sports increase the risk, if you did not have the doctor's permission before; there is also the attitude to avoid or not some dangers, I think for example to the driving style. All these factors intuitively linked to life expectation.

**Bruno:** We do not neglect the results of a check-up, often required to stipulate the insurance contract.

**Nottola:** Right!



**Bruno:** I would like to note that the *objective* risk factors, objectively to take into account to have an *objective* valuation of the *objective* probability, have the disadvantage to empty the groups to calculate the frequencies. If the initial population of age  $x$  count  $N=100.000$  individuals, distinguishing by sex you could have for example  $N_M=40.000$  males and  $N_F=60.000$  females. If we add marital status, the subpopulation of the males could split in  $N_{MS}=15.000$  married and  $N_{MC}=25.000$  bachelors, and instead for female we could have  $N_{FS} = 35.000$  married and  $N_{FN} = 25.000$  unmarried. If we introduce other risk factors, then our groups will become gradually poor, with few observations to calculate the frequency, and as consequence our empirical estimation will be unsatisfactory.

**Nottola:** I glimpse a merciless conclusion ...

**Bruno:** If you are hunting the ghost of the real probability, if you chase the Morgana Fairy of the objective probability, then you will end up accommodating so many risk factors that the observation will remain only one, no statistics will be possible and you will be forced to a personal evaluation. To get the highest possible level of objectivity, you will discover to be subjectivists!

**Chiara:** You had to stop before, of course. You have to stop the classification process *at the right point*.

**Bruno:** Who decides that? On what basis? Why should be included some risk factors and other not?

**Chiara:** It is necessary to take into account *the narrower class for which we are still able to determine the probability*.

**Bruno:** There climbs on the words, in a desperate tentative to give meaning to what has none. We passed one after the other the objective probability, the frequency, the right point of arrest and narrower class. At each request to clarify a concept from an operational point of view, you propose some other indefinite concepts, in the tacit hope that from a stream of words will emerge, by parthenogenesis, a logical proposition or a mathematical formula or even just a rule of thumb, or simply that your counterparty will decide to abandon the discussion, for exhaustion. The difficulties to objectify the probability, as opposed to the Guard of Napoleon, always retreat, but they never die. Each definition apparently objective requires subjective judgments, which can be hidden, not deleted. Of course, you can make all perfectly logical, if you remain in silence about your judgments, you can renamed them as hypothesis, but the value of these

assumptions depended on personal judgments, and it is well that we must talk of these as essential fact.

**Nottola:** Also the non-life insurance companies are in the same condition?

**Bruno:** Obvious. If you want to say “the annual frequency of the car accidents is 5%” – and then you want to use 5% as probability, to conjecture the number of the future incidents - will be necessary to bring together the individual drivers into classes, so that you can count the number of claims, to have the frequency. In which way will you build your classes? By city? By age of the driver? By sex? By number of past incidents? By type of the car? Or through a combination of some or all of these and other variables? Each class will have its own frequency - so its probability, admitted to want to assimilate frequency and probability - so each driver will have one, none and hundred thousand probabilities to have an accident, as many as the possible classes where you can place him.

**Nottola:** Can we apply the same point of view to the banks?

**Bruno:** It cannot be otherwise and I would like to expand the perspective of our analysis. I read from a book - written over thirty years ago and its author would then become a central banker - and I want to emphasize some passages: “A mathematical model, even in its most refined details, can provide support for the decision within limits. These limits exclude that the decision runs arbitrariness. Beyond those limits, the decision is left to the appreciation of discretionary individual cases based on the banker's experience and judgment. The final choice is therefore necessarily subjective. The probabilities of various outcomes of the loan, taking into account all the information that is useful acquire about the company to be given, shall be assessed only subjectively. Even when we rely on a mathematical model, an art element cannot be eliminated, at least in the estimations for the values of the exogenous variables on which each model is based. As consequence, the same loan, convenient for a bank, may look to other unacceptable for other banks for several reasons: different assessment of the probabilities of the individual outcomes, different composition of the credit portfolio, the different degree of risk aversion, different ability to deal with the risk”

**Nottola:** ... *subjective choices, ... subjective evaluations, ... unavoidable art elements, ... different probability assessments, ...* I do not understand. The mathematical models, with all their paraphernalia of formulas, data and calculations, have the aim to ensure objectivity in the risk measurement and standardize the decision processes. Or not?

**Bruno:** The outcome of a model is a number, not a decision. Models do not objectify anything. Their aim is rather to declare, clarify, explain the unavoidable elements of subjectivity, required to reason and decide under uncertainty conditions. They ensure consistency in the evaluation and make them transparent to third parties, if technically equipped. With a model is easier to formalize a coherent opinion and realize a comparison between different viewpoints.

**Nottola:** The most recent proposals of the Basel Committee on the regulation of internal rating systems, however, seem to mark a deep difference between two kinds of situations. On one side, we have the situations for which there is a huge of statistic information, and on the other side, we face instead situations where the empirical evidence is almost absent. We can think at the contraposition between high default and low default portfolios. For the residential mortgages, for example, we have always a well-defined set of historical default rates, high or low depending on the business cycle, and we can perform a reliable estimation of the probability. However, for other portfolios, for example large corporate, banks and sovereign, the default event is an “isolated” event, sometimes completely absent, and then no risk estimation is possible.

**Chiara:** Exactly! This distinction – between high default and low default – is related to the contraposition between risk and uncertainty. We must separate the concept of risk from the uncertainty. We must distinguish between the peculiar uncertain situations where there is a large agreement of the opinions and therefore easier possibilities of measurement and control, and the situations for which the lack of shared information does not permit a numerical evaluation, and so we can only undergo or avoid the effects, or face them with a purely conventional style.

**Bruno:** This discussion about risk and uncertainty – initiated by Keynes and Knight in the 30s of last century – can now be considered resolved in the foundations of the game theory. A criterion of well-defined decision should have general validity, not being deducted from specific empirical hypotheses, but from general and simple logical consistency conditions. It makes no sense to restrict the validity of the criterion to cases of "risk" based on assumptions unnecessary and ill-defined, leaving the other cases without a criterion or with an arbitrary number of more or less artificial solutions. Moreover, if you define as "risky" the situations involving minor differences from one individual to another, your definition is weak, because for judgments is always a matter of degree. Therefore, the distinction between

risk and uncertainty, in the sense proposed by Keynes and Knight, is not expressive. The greatest danger is to make her feel net and fundamental instead nuanced and ancillary.

**Chiara:** The contrast between risk and uncertainty is aligned with the intuition. We feel indeed a difference between a bet on “head and cross” and a bet on a football match and we require different odds, to participate at the game; in the guise of an insurer, we will fix different contractual clauses to take in our portfolio the risk of a car accident or an earthquake.

**Bruno:** No doubt. These moods are rooted in common feeling and I do not want to put them into question. I say only to pay attention to the danger of misunderstandings. You make a bad mistake, if you believe that the differences in uncertain situations have the power to limit the scope of the probability theory, whereas they are only linked to the outward items, which could have a role to address the evaluations and behaviour of each us, without affecting the validity of probability theory, its universal applicability. The situations of uncertainty differ only in degree, not by nature, and the probability theory is a unique and general method to rule the uncertainty, it is a unitary theory that welcomes in a general framework the different partial views, without forcing them.

**Chiara:** I’m not convinced at all. When you have a large number of observations - even if they were merged into classes in a subjective way, even if they were organized into groups based on personal judgments – we are in an environment where a mass effect works. Otherwise, without a pile of observations, we have only individual effects and of course we can’t rule them. Thus, the measurement strictly depends on the existence of a pile: as long as you have a few observations, they do not constitute a heap and we can conclude nothing, but if there is a pile, then, but only then, we have some hope to perform a reasonable measure.

**Bruno:** Let me understand: if you plan on adding a remark at a time, nothing could be said until their number is insufficient to form a pile, but the conclusion will leap out, finally, when the *no-pile* will be turned into a pile. Suddenly? Going from 99 to 100? O from 999 to 1.000?

**Chiara:** This is a caricature. Of course there is not a sharp jump ...

**Bruno:** ... of course! The no-pile passes through a to-be-or-not-to-be-an-pile phase, inclining first one way and then the other, and only subsequently does it gradually transform itself into a real and genuine aggregate ...

**Nottola:** Even more grotesque and vaguely ironic ...

**Bruno:** Inevitably! Because we do not answer the original objection raised against the distinction, here put forward as being of fundamental conceptual importance, between the pile effect and the effect of individual elements.

**Nottola:** How do we come out?

**Bruno:** We must deny any such distinction. The conclusion which can be reached on the basis of a mass of data is determined not globally, as a mass effect, but as cumulative effect of the contribution of every single information. I do not want to hide the differences between situations of uncertainty objectively different. Simply I do not want to dramatize them, creating an artificial classification, which leads towards inadequate and distorted interpretations. In general, the relevant information required to fix an opinion could be different – from individual to individual, from bank to bank – and depending on the circumstances the opinions – the probabilities – will have different degrees of stability and sharing. Some of them will be well established and widely accepted, other will be more undulating, others will remain interim, but they are all opinions and they remain opinions. We can also make a distinctions – for example high default versus low default – if we think that they are useful, but these distinctions concern only external aspects of the available information, and do not affect the unitary nature of the probability and its application in all fields. On the other hand, when a banker lends money, when he makes a price for the loan, when he draws up the financial statement or makes a decision about capital level, he is assigning a probability, even if he is not aware of it, and the banker must do all this regardless if his counterparty is into a high or low default portfolio. The refuse to evaluate a probability, on the pretext that there is not enough empirical evidence, it means acting like Cremonini, scientist contemporary of Galileo, which, with some folklore, it is said he did not want to look through the telescope to avoid to change his beliefs about the cosmos.

**Nottola:** Excuse me, but in all that, what happened to the axiomatic theory, which gives a *mathematical definition* of probability?

**Bruno:** The axioms and theorems of the probability theory fix only a huge of constraints between individual evaluations, totally free in all other respects. Given two events  $A$  and  $B$  – prepositions for which we will be able to say if they will be true or false – a classic theorem of probability theory is written in the form:

$$P_1 = (A \cup B|H) = P_1(A|H) + P_1(B|H) - P_1(A \cap B|H)$$

This means: “the individual I, with his information set H, must assess the probability of the occurrence of at least one event between *A* and *B* so that its assessment equals the sum of the probabilities of the individual events, minus the probability to have *A* and *B* together”. Mathematical probability theory shields your opinion from the contradictions as well as the geometry ensures the consistency in the measures of the physical objects. If anyone, after measuring a rectangle, said he had found a base of 5 cm, a height of 10 cm and an area of 25 cm<sup>2</sup>, we can say to him to revise at least one measurement, without discussion on which measure, because it is not true that  $5 \times 10 = 25$ . The axiomatic theory only state the constraint  $\mathbf{P}_I = (A \cup B | \mathbf{H}) = \mathbf{P}_I(A | \mathbf{H}) + \mathbf{P}_I(B | \mathbf{H}) - \mathbf{P}_I(A \cap B | \mathbf{H})$ , but it does not give you an instrument to evaluate the single pieces of the formula. Therefore, we have an infinity of assessments itself coherent, because they respect the constraint of consistency. Everyone will choose what he prefers or, to be right, one will choose what he feels. In conclusion, the models are useful, applicable everywhere and appropriately, provided that each one always start from his opinions; but the models become illusory, and therefore dangerous, if you plan to use them in a miraculous way, to avoid to declare your view, thinking to set a tool to create the opinions, whereas it can only to manipulate them.

**Nottola:** To return on the credit risk of a counterparty ...

**Bruno:** ... the same loan is otherwise risky if placed in a different portfolio, if it is considered from different banks.

**Nottola:** I already see a theatrical grimace of disappointment, on the face of regulators and supervisors.

**Bruno:** I am sorry, but there is no way in which the individual can avoid the burden of responsibility for his own evaluations. The key cannot be found that will unlock the enchanted garden wherein, among the fairy rings and the shrubs of magic wands, beneath the trees laden with monads and noumena, blossom forth the flowers of *Probabilitas realis*. With these fabulous blooms safely in our button-holes we would be spared the necessity of forming opinions, and the heavy loads we bear upon our necks would be rendered superfluous once and for all.

**Chiara:** However, I continue to see a link between probability and frequency similar to the linkage between the true measure of a physical quantity and its experimental evaluations affected by errors due to the measuring instrument.

**Bruno:** The analogy is pushed beyond the reasonable, because forgets – to make a concession of jargon and mind-set – that the probability is an atypical measure of an invisible object. The reliability of a frequency for the evaluation of a probability depends on your beliefs. It make no sense to say “probability” instead of “frequency”, thinking to make objective the probability, if the connection between the technicalities to detect the frequency (in the past) and the way to have an opinion (about the future) remains subjective. It is better analyse directly the subjective element in which is rooted the notion of probability.

**Nottola:** Anyway, there is a strong psychological reluctance to abandon the idea of objective probability.

**Bruno:** We are unfortunately under the *tyranny of language*. The word *subjective* evokes the most bizarre and capricious attitudes, whereas objective would means a serious and reasonable behaviour. The hesitation to accept the subjective probability, in the sciences and applications, is due to the impression that if everything is subjective, then everything has to be arbitrary and no rule can enforce. But in its technical meaning, free from emotional influences, subjective simply means *related to a subject* and subjective assessments are estimates made by a person, by an individual, who takes responsibility and accepts the consequences. Subjective is all that requires intervention from the judgment of an individual, because you cannot reduce it to the mere finding of a fact. The views of a mathematician and of a cabbalist, of an astronomer and of an astrologer, of a doctor and of a self-styled healer, are certainly subjective, equally subjective because all expressed by individuals, and you could grant them more or less confidence. The subjective probabilities school does not want to accept superstitious or metaphysical interpretations, only it keen to emphasize the need to recognize, in all reasoning, what is logical, what has an empirical meaning and what is a pure subjective value, because this distinction should be established in any mathematical theory to be able to usefully deepen criticism of principles.

**Nottola:** It is a revolutionary change!

**Bruno:** I understand very well the difficulties that those who have been brought up on the objectivistic conceptions meet in escaping from them. I understand it because I myself was perplexed for quite a while some time ago. It was only after having analysed and mulled over the objectivistic conceptions in all possible ways that I arrived, instead, at the firm conviction that they were all irredeemably illusory. It was only after having gone over

the finer details and developed, to an extent, the subjectivist conception, assuring myself that it accounted for everything that is usually accredited, over hastily, to the fruit of the objectivistic conception, it was only after this difficult and deep work, that I convinced myself, and everything became clear to me. It is certainly possible that these conclusions are wrong; in any case they are undoubtedly open to discussion, and I'm glad we discussed it.

**Chiara:** However, my point of view is paradoxically strengthened precisely by the arguments of my counterparty. If he proclaims the subjectivity of probability of default, then I do not understand how you can think to build a banking regulation on this basis. The rules would be based on the *soft mud of the feelings*, the entire institutional architecture would be built on *the sand of the internal models*.

**Bruno:** If you remove the sand, then you will build on the empty. The regret of losing faith in the objectivity of probabilities, and in the regulatory measures that use it, it is unjustified. Nothing is lost, except what was an illusion. However, because everything in life is uncertain, and outside of that, nothing can assert with certainty, I do not want to preclude the possibility to change my idea. Therefore, if a day you you'll have access to the strongbox of the Platonic Truths, and you will explore it to see if there are or not the *true probabilities* that you are looking for, I promise to adopt them, if you will find them. Until then, I will not care about of the objective truths, and I will work with something formally much less, but substantially much more: *my opinion*.

**Nottola:** We live in a complex world. The financial system has long globalized and now also the regulation and supervision begin to be globalized, overlapping and mixing different experiences, traditions and cultures. How can we orient ourselves in such a multifaceted world?

**Bruno:** The North Star is the commitment to be honest and earnest, against *the hundred ways to say nothing*; it is the engage to have and declare the *opinion*; it is the constraint to be *coherent*.

### 3. Conclusion and next steps

This *Dialogue* wished to put the conceptual basis for a consistent discussion about risk measurement in banking and insurance environments. We will be now committed to translate the main theses of the *Dialogue* in a technical content. The main issue for regulators and supervisors of the financial sector is double-faced: on the one hand, they want to understand



how much confidence can be granted to the risk measures proposed by the intermediaries; on the other, they want to be sure that firms have the most effective incentives for a measurement that, without pretending to be “true”, should be at least as earnest and honest as possible. The binomial test – when we treat about credit risk – is methodologically weak, with respect to these requirements. Our proposal, on the first stage, use a Bayesian “credibility” mechanism to mix the views of the regulator and the company into a single probability measure (of default) to be used to calculate capital requirements. Then we propose the use of the penalty method – “the only meaningful way to deal with the prediction problem”, to quote de Finetti – to assess the probability measures proposed by banks and push them towards the *best-effort* to have their *best-estimate*. An appropriate calibration of our proposal allow to rule in a general scheme the so-called *Trouth-the-Cycle* and *Point-in-Time* probabilities, to meet effectively the requirements of the supervisors and accounting standard setter.

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